

# Apparent motion of a spherical shell collapsing onto a black hole

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## Abstract

We model the collapse of a spherically symmetric, constant density, pressureless shell onto a preexisting black hole. Contrary to a recent claim of Liu and Zhang (2009), we show that an observer at infinity never sees any part of the shell cross the event horizon. The entire shell appears to “freeze” outside the black hole. We find that during intermediate stages of the collapse, the change in apparent size of the inner surface is non-monotonic. However, at late times, the apparent sizes of the inner and outer surfaces of the shell both approach  $3\sqrt{3}GM/c^2$  asymptotically, where  $M$  is the combined mass of the initial black hole and the shell.

*Keywords:*

Black holes, Classical theories of gravity, Space-time singularities

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## 1. Introduction

The collapse of a spherically symmetric, constant density, pressureless “star” to a black hole was solved by Oppenheimer and Snyder [1] (hereafter OS39). They showed that an observer on the star hits a singularity in finite proper time, while an observer at infinity sees the star slow down and “freeze” as it approaches the event horizon. The apparently contradictory experiences of the two observers was not reconciled until Finkelstein found a horizon-penetrating coordinate system accommodating both viewpoints [2].

The OS39 model has since been generalized in many directions. Misner and Sharp [3] discussed adiabatic collapse of an ideal fluid. Vaidya added radiation and gave non-adiabatic solutions [4]. Planar and cylindrically-symmetric collapse scenarios were considered by Liang [5]. The collapse of a slowly rotating cloud to a Kerr black hole was modeled by Kegeles [6]. Magnetized collapse was investigated numerically by Baumgarte and Shapiro [7].

None of these generalizations challenged the original conclusion that the observer at infinity sees the collapsing material freeze as it approaches the horizon. However, recently Liu and Zhang [8] (hereafter LZ09) claimed that when a finite thickness shell collapses onto a preexisting black hole, all but an infinitesimal surface layer of the shell disappears across the horizon in finite

time. Their argument rests on the construction of a time coordinate in the shell that they identify with the observer at infinity’s proper time. We disagree with this interpretation. If one computes the light travel time across the shell in their coordinates, one finds it diverges as the inner surface of the shell approaches the horizon. So no part of the shell is observed entering the hole and it freezes outside.

In addition to potential time delay as a result of gravitational redshift, there is also a geometrical time delay. Light rays in the Schwarzschild metric with impact parameters near  $D = 3\sqrt{3}M$  (where we set  $G = c = 1$ ) follow tightly wound spirals. As  $D \rightarrow 3\sqrt{3}M$ , the path length increases and the time delay diverges. So the outer edge of a collapsing star’s disk appears to freeze away from the horizon, at  $3\sqrt{3}M$  [9].

In this Letter, we give a solution for the collapse of a shell onto a preexisting black hole that is simpler than the construction of LZ09. We describe how the horizon expands with time and show that the light crossing time across the shell diverges as the material approaches the horizon. We solve for the apparent sizes of the inner and outer surfaces of the shell as a function of time. At intermediate stages of the collapse, the apparent size of the inner surface varies non-monotonically. At late times, both surfaces freeze as they approach size  $3\sqrt{3}M$ , where  $M$  is the combined mass of the initial black hole and the shell.

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## 2. Collapse solution

### 2.1. Metric in the exterior of the shell

Consider a spherical shell with negligible internal pressure which collapses onto a preexisting black hole. The geometry of the spacetime external to the shell will be the Schwarzschild geometry throughout the collapse, by Birkhoff's theorem. This is true of any spherical configuration which does not radiate an appreciable fraction of its mass. General relativity prohibits monopole gravitational waves, so there is no possible way for any gravitational influence of the radial collapse to propagate outward [10]. By the same argument, we also have the Schwarzschild geometry in the cavity between the shell and black hole. The coordinates constructed by LZ09 are not Schwarzschild, but this is an artifact of their construction and not, as they claim, a failure of Birkhoff's theorem.

### 2.2. Metric in the interior of the shell

First consider the interior of a collapsing, constant density star. The metric is a patch of the Friedmann-Robertson-Walker (FRW) metric :

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + d\chi^2 + \sin^2 \chi d\tilde{\Omega}^2 \right], \quad (1)$$

$$d\tilde{\Omega} = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2. \quad (2)$$

This is expected because the spacetime in the star is homogeneous and isotropic. It can also be derived from Einstein's equations [11]. The solution used in cosmology to describe a spatially flat universe has curvature  $K = 0$ , but the interior of the star has the spatial geometry of a 3-sphere, so  $K = 1$ . The Einstein equations demand for the scale factor

$$a(\eta) = (a_i/2)(1 + \cos \eta). \quad (3)$$

We let the time coordinate  $\eta$  run from 0 to  $\pi$ , over which period the FRW universe contracts from an initial size  $a = a_i$  to a singularity. We limit the radial coordinate to the range  $0 < \chi \leq \chi^{\text{out}}$ , and match onto the external Schwarzschild geometry at  $\chi^{\text{out}} \equiv \sin^{-1}(r_i^{\text{out}}/a_i)$ . The angular coordinates  $(\tilde{\theta}, \tilde{\phi})$  have their usual ranges and can be matched with the angular coordinates of the Schwarzschild metric at the boundary of the star. The resulting star has mass  $M = (a_i/2) \sin^3 \chi^{\text{out}}$ , initial radius  $r_i^{\text{out}} = a_i \sin \chi^{\text{out}}$ , and initial density

$$\rho_i = \frac{3}{8\pi a_i^2} = \frac{M}{4/3\pi (r_i^{\text{out}})^3}. \quad (4)$$

Now we would like to replace the star with a shell collapsing onto a preexisting black hole. We excise a ball of mass  $m$  from the center of the star and replace it with a mass  $m$  Schwarzschild black hole. To preserve the FRW metric in the shell's interior, we need the initial density of the shell to be (4), where  $M$  is now the total mass of the initial black hole and the shell. In other words, we require  $m = M \left( 1 - (r_i^{\text{in}}/r_i^{\text{out}})^3 \right)$ , where  $r_i^{\text{in}}$  is the initial radius of the inner surface of the shell. This leaves one dimensionless free parameter in the solution,  $r_i^{\text{in}}/r_i^{\text{out}}$ . The FRW metric matches to the external Schwarzschild geometry (with mass  $M$ ) at  $\chi^{\text{out}}$ , as before, and to the internal Schwarzschild geometry (with mass  $m$ ) at the inner surface of the shell at  $\chi^{\text{in}} = \sin^{-1}(r_i^{\text{in}}/a_i)$ . The FRW radial coordinate is limited to the range  $\chi^{\text{in}} \leq \chi \leq \chi^{\text{out}}$ .

### 2.3. Collapse solution in coordinates

The radial coordinates of particles falling with the shell are related to FRW coordinates by:

$$r = (r_i/2)(1 + \cos \eta). \quad (5)$$

In these coordinates, every particle in the shell hits the singularity at  $\eta = \pi$ . Figure 1 shows an example with  $r_i^{\text{out}} = 10$  and  $r_i^{\text{in}} = 7$ .

The Schwarzschild time coordinate in the region outside the outer boundary of the shell,  $t$ , is related to  $\eta$  by [10]:

$$t = 2M \log \left[ \frac{(r_i^{\text{out}}/2M - 1)^{1/2} + \tan(\eta/2)}{(r_i^{\text{out}}/2M - 1)^{1/2} - \tan(\eta/2)} \right] + 2M (r_i^{\text{out}}/2M - 1)^{1/2} \left[ \eta + (r_i^{\text{out}}/4M)(\eta + \sin \eta) \right]. \quad (6)$$

Equations (5) and (6), evaluated at  $r = r^{\text{out}}$ , describe radial free fall in the mass  $M$  Schwarzschild metric.

The Schwarzschild time coordinate in the cavity between the initial black hole and the shell,  $t'$ , is not the same as  $t$ :

$$t' = 2m \log \left[ \frac{(r_i^{\text{in}}/2m - 1)^{1/2} + \tan(\eta/2)}{(r_i^{\text{in}}/2m - 1)^{1/2} - \tan(\eta/2)} \right] + 2m (r_i^{\text{in}}/2m - 1)^{1/2} \left[ \eta + (r_i^{\text{in}}/4m)(\eta + \sin \eta) \right]. \quad (7)$$

The inner edge of the shell falls freely in the mass  $m$  Schwarzschild metric.

#### 2.4. The event horizon

The outer boundary of the shell collapses across  $r = 2M$  at

$$\eta_* = \cos^{-1} \left( \frac{4M}{r_i^{\text{out}}} - 1 \right), \quad (8)$$

as follows from (5). The event horizon is generated by the null geodesic separating events which can send signals to infinity from those that cannot. At late times,  $\eta \geq \eta_*$ , the horizon is fixed at  $r = 2M$ . The location of the event horizon at earlier times can be found by tracing this geodesic backwards<sup>1</sup>.

In the FRW patch in the interior of the shell, outgoing radial null geodesics ( $ds^2 = d\Omega = 0$ ) satisfy:

$$\frac{d\chi}{d\eta} = +1, \quad (9)$$

so the solution for the horizon position in the interior of the shell is

$$\chi_H = \sin^{-1} \left( r_i^{\text{out}}/a_i \right) - (\eta_* - \eta). \quad (10)$$

The horizon meets the inner boundary of the shell at

$$\eta_{**} = \eta_* - \Delta\eta, \quad (11)$$

$$\Delta\eta \equiv \sin^{-1} \left( r_i^{\text{out}}/a_i \right) - \sin^{-1} \left( r_i^{\text{in}}/a_i \right). \quad (12)$$

It can be extended into the infinite past according to

$$\frac{dr}{dt'} = 1 - \frac{2m}{r}, \quad (13)$$

which is the null geodesic equation in the mass  $m$  Schwarzschild geometry governing the cavity between the initial black hole and shell. The horizon position,  $r_H(\eta)$ , is given implicitly by:

$$t'(\eta) = r_H + 2m \log(r_H - 2m) + C, \quad (14)$$

$$C \equiv t'(\eta_{**}) - (r^{\text{in}}(\eta_{**}) + 2m \log(r^{\text{in}}(\eta_{**}) - 2m)). \quad (15)$$

In this equation,  $t'(\eta)$  is given by (7). To summarize: at late times, the horizon is fixed at  $r = 2M$  outside the outer surface of the shell. At intermediate times, it expands in the interior of the shell according to (10) and, at early times, it expands in the cavity between the black hole and shell according to (14).

Figure 1 shows the expanding horizon of a collapsing shell. Note that the horizon begins expanding before any material has crossed  $r = 2m$ , and it is at

<sup>1</sup>LZ09 mislabel this surface as the “apparent horizon.” The outermost apparent horizon is at  $r = 2m$  for  $\eta < \eta_*$  and it is at  $r = 2M$  for  $\eta > \eta_*$ .

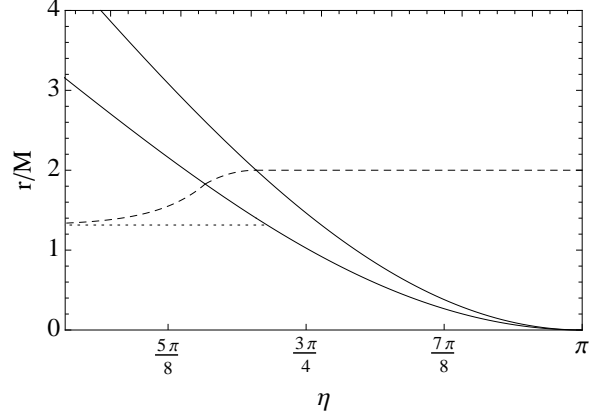


Figure 1: Collapse solution as a function of the time-like coordinate  $\eta$  for  $M = 1$ ,  $m = 0.657$ ,  $r_i^{\text{out}} = 10$ , and  $r_i^{\text{in}} = 7$ . The inner and outer surfaces of the shell (solid curves) collapse to a singularity at  $\eta = \pi$ . The event horizon (dashed) approaches  $r = 2m$  in the infinite past and grows to  $r = 2M$  as the shell collapses. Note that the horizon begins expanding before any material has crossed  $r = 2m$  (dotted).

$r > 2m$  in the cavity between the black hole and the shell, even though the geometry is Schwarzschild with mass  $m$  there. This illustrates the fact that event horizons depend on the complete history of a spacetime and cannot be detected by local measurements. This is the teleological property of horizons.

Finally, consider the time,  $t_c$ , that it takes photons emitted at the inner surface of the shell at time  $\eta_e$  to cross the shell:

$$t_c = t(\eta_e + \Delta\eta) - t(\eta_e). \quad (16)$$

As the point of emission approaches the horizon,  $\eta_e \rightarrow \eta_{**}$ , the crossing time  $t_c \rightarrow t(\eta_*) - t(\eta_{**})$ . The first term corresponds to  $r \rightarrow 2M$  in (6), which diverges. So an observer at infinity never sees any part of the shell disappear behind the horizon.

#### 2.5. Apparent motion of the outer surface

So far we have only considered radially emitted photons. A photon reaching the observer appears a distance  $D = u_\phi/u_t$  from the center of the shell, where  $u_\mu$  is the photon's 4-momentum and  $D$  is its impact parameter. Radially emitted photons have  $D = 0$  and appear at the center of the collapsing shell. The apparent radius of the shell is the largest impact parameter of the arriving photons. So we need to consider non-radial photon trajectories.

At each point on the shell, photons are emitted with a range of impact parameters. The photon with the largest impact parameter follows the null geodesic of the Schwarzschild metric that is tangent to the surface of the

shell. This photon has  $u_\mu u^\mu = g^{\phi\phi} (u_\phi)^2 + g^{tt} (u_t)^2 = 0$ , or

$$D = \sqrt{\frac{g_{\phi\phi}}{-g_{tt}}} = \frac{r^{\text{out}}}{\sqrt{1 - 2M/r^{\text{out}}}}. \quad (17)$$

This photon escapes to infinity if  $r^{\text{out}} \geq 3M$  [12]. We do not consider the case  $r^{\text{out}} \leq 3M$  here because we are going to find that the apparent size of the outer surface freezes as  $r^{\text{out}} \rightarrow 3M$ .

When  $r^{\text{out}} \gg 3M$ , we find  $D \approx r^{\text{out}}$ , the usual flat space result. As the shell approaches  $3M$ , its apparent size is given by (17). Spacetime curvature increases the time delay between the emission and arrival of photons, but for simplicity we can ignore this in the early stages of collapse and use the  $r^{\text{out}}$ -motion given by (5)-(6).

Geometric time delay begins to dominate the appearance of the collapse near

$$D_c \equiv 3\sqrt{3}M \approx 5.196M. \quad (18)$$

Photons with  $D \approx D_c$  spend a long period of coordinate time on tight, outgoing spirals before escaping to infinity. As the outer surface of the shell passes  $r = 3M$ , it leaves behind a ‘photon cloud’ whose apparent size shrinks to  $D_c$  asymptotically. Near  $D_c$ , the apparent collapse can be approximated by an exponential with e-folding time  $2/(3\sqrt{3}M)$  [9]:

$$D = D_c + (D_1 - D_c) \exp\left(-2\frac{t - t_1}{3\sqrt{3}M}\right), \quad t > t_1, \quad (19)$$

where  $D_1$  is a constant and  $t_1$  is the coordinate time at which the apparent size of the shell is  $D_1$ . This formula is only strictly valid near the critical impact parameter,  $D_1 \rightarrow D_c$ . To an observer at infinity, the outer surface of the shell appears to freeze as it approaches  $D_c$ .

In the early stages of collapse we ignore geometrical time delay and use (17) with  $r^{\text{out}}$  given by (5)-(6). But in the late stages of collapse, we will use (19) with

$$D_1 = D_c + 648\sqrt{3} \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)^2} \exp(-3\pi) M \approx 5.203M, \quad (20)$$

which is the largest impact parameter for which a photon can complete a single loop around the black hole [12].

Figure 2 shows how the apparent size of the outer surface asymptotically shrinks to  $3\sqrt{3}M$  as a shell collapses.

## 2.6. Apparent motion of the inner surface

Now consider photons which leave the inner surface of the shell and propagate outwards. As they pass

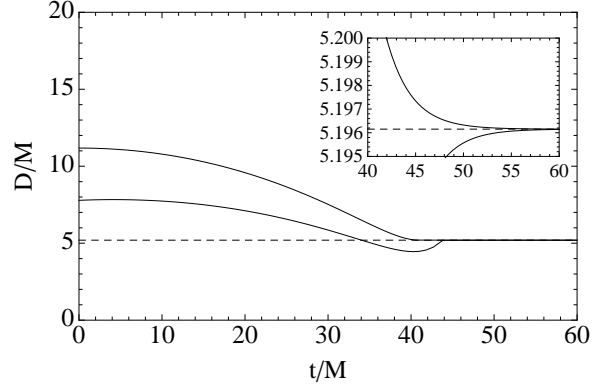


Figure 2: Apparent sizes of the inner and outer surfaces of a collapsing shell (solid curves) as a function of the observer at infinity's proper time,  $t$ . The collapse parameters are as in Figure 1. At intermediate times, the apparent size of the inner surface varies non-monotonically. At late times, both surfaces approach  $3\sqrt{3}M$  asymptotically (dashed).

through the shell they follow geodesics of the FRW metric. When they reach the outer surface of the shell they enter the external Schwarzschild geometry and continue out to infinity. We are going to find it convenient to replace the FRW time coordinate  $\eta$  with a comoving time coordinate  $\tau$  defined by  $d\tau^2 = a^2 d\eta^2$ . Now  $g_{\tau\tau} = -1$ , and photon geodesics satisfy [11]

$$u_\tau = -u^r = a. \quad (21)$$

$D$  is not conserved as photons travel across the shell because the FRW metric (1) is time-dependent. Let  $p_1$  be the spacetime event at which a photon is emitted at the inner surface of the shell and let  $p_2$  be the spacetime event at which it arrives at the outer surface. We need

$$D = \frac{u_\phi|_{p_2}}{u_t|_{p_2}}, \quad (22)$$

the impact parameter of the photon as it leaves the shell. This will be the impact parameter of the photon at infinity.

Matching coordinates across the outer boundary of the shell gives

$$u_\phi|_{p_2} = \left(\frac{r^{\text{out}}}{a \sin \chi}\right)_{p_2} u_\phi|_{p_1}, \quad (23)$$

where we have used the fact that  $u_\phi$  is conserved along FRW geodesics to set  $u_\phi|_{p_2} = u_\phi|_{p_1}$ . We maximize  $u_\phi|_{p_1}$  by considering the photon emitted tangentially at  $p_1$ . This gives  $u_\phi|_{p_1} = \sqrt{g_{\phi\phi}} u^r$  or

$$u_\phi|_{p_2} = \left(\frac{r^{\text{out}}}{a \sin \chi}\right)_{p_2} (\sin \chi a^2)_{p_1}. \quad (24)$$

The energy at infinity is

$$u_t|_{p_2} = \sqrt{1 - 2M/r^{\text{out}}} u_\tau|_{p_2} = \sqrt{1 - 2M/r^{\text{out}}} a. \quad (25)$$

In the first step we matched coordinates across the outer boundary of the shell and in the second step we used (21).

Combining (22), (24), and (25) gives the apparent size of the inner edge of the shell:

$$D = \frac{r^{\text{in}}(\eta - \Delta\eta)}{\sqrt{1 - 2M/r^{\text{out}}(\eta)}}. \quad (26)$$

The shell crossing time,  $\Delta\eta$ , should be evaluated for geodesics emitted tangentially to the inner surface of the shell. However, for simplicity we will continue to use the radial time delay (12).

In the limit of vanishing shell thickness,  $\Delta\eta \rightarrow 0$ , equation (26) reduces to (17) and the apparent sizes of the inner and outer surfaces of the shell are the same.

Figure 2 shows the non-monotonicity of the apparent size of the inner surface of the shell as it approaches  $3\sqrt{3}M$ .

### 3. Conclusions

None of these results will be observable in the foreseeable future, as the intensity of a star collapsing to a black hole is attenuated exponentially on a timescale of order the black hole light crossing time [13]. However, the analysis has clarified several features of gravitational collapse and horizons.

The observer at infinity does not see any part of the shell cross the event horizon. It freezes entirely outside. We have checked that the light-crossing time across the shell diverges as the shell approaches the horizon. This was shown for the coordinates constructed in this paper and we have checked it is true also for the coordinates used by LZ09.

We investigated the apparent sizes of the inner and outer surfaces of the shell during collapse. During intermediate stages, the apparent size of the inner surface varies non-monotonically, an effect which is absent from the collapse of an infinitesimally thin shell, or from the apparent collapse of the surface of a star. The apparent sizes of the inner and outer surfaces of our shell both asymptote to  $3\sqrt{3}M$  at late times.

We fixed  $m/M = 1 - (r^{\text{in}}/r^{\text{out}})^3$  for convenience, so that the metric inside the shell would be FRW. However, the qualitative results are valid more generally, for generic values of  $m/M$ . Our analysis applies equally

well to intermediate layers in the shell or in the collapse of a homogeneous star.

The collapse of multiple shells onto a preexisting black hole is qualitatively similar. Birkhoff's theorem implies the Schwarzschild metric is valid outside the shells. For certain choices of the shells' densities, FRW patches describe the spacetime inside the shells. The horizon now depends on the number of shells and their relative sizes, but it can be located by tracing backwards from the outermost shell as in §2.4. All of the infalling material asymptotes to  $3\sqrt{3}M$ , where  $M$  is now the total mass of the initial black hole and all of the shells. The approach to  $3\sqrt{3}M$  is not monotonic for the inner layers. An observer at infinity sees everything freeze outside the horizon.

Our discussion has been entirely classical. A quantum mechanical treatment of spherical collapse is possible when the black hole mass is much larger than the Planck mass and one again finds that collapsing material does not appear to cross the horizon in finite time [14]. This rule might be violated by Planck mass black holes, for which quantum fluctuations of the metric are important [15]. However a consideration of the black hole information problem suggests a deeper, black hole ‘‘complementarity’’ principle may be at work, and complete information about collapse always remains available outside horizons [16, 17].

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